Refined Razumov-Stroganov conjectures for open boundaries

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Abstract

Recently it has been conjectured that the ground-state of a Markovian Hamiltonian, with one boundary operator, acting in a link pattern space is related to vertically and horizontally symmetric alternating-sign matrices (equivalently fully-packed loop configurations (FPL) on a grid with special boundaries). We extend this conjecture by introducing an arbitrary boundary parameter. We show that the parameter dependent ground state is related to refined vertically symmetric alternating-sign matrices i.e. with prescribed configurations (respectively, prescribed FPL configurations) in the next to central row.

We also conjecture a relation between the ground-state of a Markovian Hamiltonian with two boundary operators and arbitrary coefficients and some doubly refined (dependence on two parameters) FPL configurations. Our conjectures might be useful in the study of ground-states of the O(1) and XXZ models, as well as the stationary states of Raise and Peel models.

1 Introduction

In a remarkable paper, Razumov and Stroganov (R-S) [1] have looked at the ground-state wave function of the ferromagnetic one-dimensional XXZ spin 1/2 chain with the asymmetry parameter $\Delta = -1/2$, odd number of sites and periodic boundary condition. They noticed that for a small number of lattice sites $L = 2n + 1$ (this was all numerics), the largest component and the normalization are related to the number of $n \times n$ alternating-sign matrices (ASMs) [2]. They conjectured that these coincidences are valid for any number of sites. This conjecture, though not yet proven, triggered a lot of other conjectures which brought together the study of quantum chains and combinatorics.

It was realized in [3] that similar conjectures can be made in the O(1) loop model with various boundary conditions. This led R-S [4], using the bijection between ASMs and fully packed loops (FPLs) on a grid with special boundary conditions, to a new conjecture (called hereafter the R-S conjecture) which gives a much deeper connection between the ground-state wave function and ASMs. Conjectures similar to the one of R-S were also made for Hamiltonians with different boundary conditions [5, 6] relating them to different symmetry classes of ASMs.

In another development (relevant to the present paper), it was understood that the results obtained in the O(1) model can be derived writing the Hamiltonians in terms of generators of the Temperley-Lieb algebra at the semi-group point [6], acting on the link pattern (or equivalently the restricted solid on solid path) basis. Each Hamiltonian gives the Markovian time evolution of a fluctuating interface. The class of stochastic models of this type are called Raise and Peel (RP) models (see [7] for physical properties and [8] for several examples). Besides their interesting properties, these models belong to a new universality class for non-equilibrium phenomena, since the finite-size scaling spectrum of these models are given by conformal field theory.

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In a very recent new development, Di Francesco [9] proposed a refined R-S conjecture. If one considers $A_n(j)$ the known number [10] of $n \times n$ ASMs with a 1 on top of the jth column one can define a generating function $\psi_n(t)$. According to Di Francesco's conjecture this function coincides with the normalization of the ground-state wavefunction of the t dependent monodromy matrix. As for the other cases when R-S type conjectures were made, not only the normalization, but each component of the t-dependent ground-state gives information on properties of ASMs.

In this paper we will present two new refined R-S type conjectures. In Section 2 we present the first conjecture. We define a Hamiltonian in terms of the L generators of the boundary Temperley-Lieb algebra. This Hamiltonian acts in the space of 2^L link patterns and depends on a free parameter a. If this parameter is nonnegative, the Hamiltonian gives the time evolution of a fluctuating interface. The parameter a controls the boundary rates. We have studied the ground-state wave function, which gives the probability distribution function of the stationary state of the stochastic process, as a function of a.

For $a = 1$, it was observed in [11], a la Razumov and Stroganov, that the ground-state wave function $\psi_0(1)$ for a system of size L is related to vertically and horizontally symmetric alternating sign matrices (VHASMs) of size $2L+3$. In the present paper we study $\psi_0(a)$ and make a refined R-S conjecture. As an application, we give the average number of ± 1's on the $(L+1)$-st row of a VHASM matrix of size $2L+3$. Other results can also be used to study the properties of stationary states of the stochastic process.

Although, it is not the main subject of this paper, we also comment on the connection of the Hamiltonian with the XXZ quantum chain with diagonal boundary conditions. We furthermore briefly discuss the properties of the spectrum of the Hamiltonian in the continuum limit.

The second conjecture is presented in Section 3. We add to the Hamiltonian considered in Section 2 a second boundary operator with a coefficient b. The new Hamiltonian depends therefore on two parameters. The space of link patterns in which the new Hamitonian acts is different from the one in which the Hamiltonian with only one generator acts. Remarkably, the dimensions of the subspaces in which the two ground-states have components are the same.

We conjecture that the ground-state $\psi_0(a, b)$ of the Hamiltonian with $L-1$ bulk and 2 boundary generators is related to weighted FPL diagrams on a grid of dimension $(L+1) \times L$ with special boundary conditions. A given configuration gets a factor a for each vertical segment at the boundary and a factor b for each horizontal segment at the boundary.

2 The refined one-boundary R-S conjecture

The Hamiltonian we wish to study is given by

$$H = a(1 - f_-) + \sum_{j=1}^{L-1} (1 - e_j), \tag{1}$$

where the generators f_- and e_i satisfy the one-boundary Temperley-Lieb algebra, otherwise known as the blob algebra [12],

$$\begin{aligned} e_i^2 &= e_i, \\ e_i e_{i\pm 1} e_i &= e_i, \\ e_i e_j &= e_j e_i \quad \text{for} \quad |i-j| \geq 2, \\ f_-^2 &= f_-, \\ e_1 f_- e_1 &= e_1. \end{aligned} \tag{2}$$

The Hamiltonian given by (1) describes the time evolution of a stochastic process of a fluctuating interface with a boundary, see [8] for the case $a = 1$. Here we are interested in the stationary state only.

The Temperley-Lieb algebra has a well known loop representation in the space of link patterns [13, 14]. Let us briefly recall this representation. Consider an $L \times 2$ strip whose L bottom sites

may be connected to each other or to the site directly above it by non-crossing arcs. In the case of the one-boundary Temperley-Lieb algebra, sites can also be connected to the left boundary of the strip. See Fig. 1 for an example of an 8×2 strip. We call a particular way in which loops are connected a *link pattern* or *connectivity*, and denote the linear span of link patterns on L sites by LP_L.

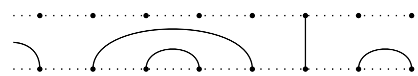


Figure 1: An 8×2 strip with link pattern)(())|().

On a link pattern $\pi \in \mathrm{LP}_L$ the generator e_i acts in the following way: If site i is connected to k and site $i+1$ to l, e_i connects i with $i+1$ and k with l. If i is connected to k and $i+1$ to the top row (or left boundary) of the strip, then k gets connected to the top row (or left boundary) and i to $i+1$; similarly, if i is connected to the top row (or left boundary) and $i+1$ is connected to k. Lastly, if both i and $i+1$ are connected to the top row (or left boundary), they get connected to each other. If site 1 is connected to i, the generator f_- acts by connecting both site 1 and site i to the left boundary of the strip. If site 1 is connected to the top row it gets connected to the left boundary, while if site 1 were connected to the left boundary, f_- acts as the identity.

Link patterns can be described using a parentheses notation: If site i is connected to site j we put an opening parenthesis "(" at i and a closing parenthesis ")" at j. If site i is connected to the left boundary we put a closing parenthesis at i. Sites that are connected to the top of the strip are denoted by vertical bars. The link patterns for $L = 2$ are thus given by

$$||, \quad)|, \quad)), \quad (), \tag{3}$$

which respectively mean that (i) the two sites are connected to the top of the strip, (ii) the first is connected to the left boundary while the second is connected to the top, (iii) both are connected to the left boundary and (iv) the two sites are connected to each other. The dimension of the space of link patterns for the one-boundary Temperley-Lieb algebra on L sites is

$$\dim \mathrm{LP}_L = 2^L. \tag{4}$$

Due to the semi-group structure of the algebra (2), the Hamiltonian (1) has a positive spectrum and a unique ground-state energy $E_0 = 0$ in LP_L. We will be interested in the corresponding eigenvector ψ_0 as a function of the parameter a,

$$H\psi_0(a) = 0. \tag{5}$$

This eigenvector lies in the subspace LP_L^0 spanned by the link patterns without vertical bars. Its dimension is equal to

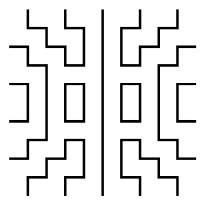
$$\dim \mathrm{LP}_L^0 = \binom{L}{\lfloor L/2 \rfloor}. \tag{6}$$

2.1 Fully packed loops

In [8, 11] the Hamiltonian (1) was studied for $a = 1$ and it was observed that $\psi_0(1)$ for size L is related to vertically and horizontally symmetric alternating-sign matrices (VHASMs). In particular it was conjectured that the sum of the components of $\psi_0(1)$ for size L is equal to $A_{\mathrm{VH}}(2L+3)$, the total number of VHASMs of size $2L+3$. This number is known [15] and given by

$$\begin{aligned} A_{\mathrm{VH}}(4n \pm 1) &= A_{\mathrm{V}}(2n \pm 1) N_8(2n), &(7)\\ A_{\mathrm{V}}(2n+1) &= \prod_{k=0}^{n-1} (3k+2) \frac{(6k+3)!(2k+1)!}{(4k+2)!(4k+3)!} = 1, 3, 26, 646, \ldots, &(8)\\ N_8(2n) &= \prod_{k=0}^{n-1} (3k+1) \frac{(6k)!(2k)!}{(4k)!(4k+1)!} = 1, 2, 11, 170, \ldots, &(9) \end{aligned}$$

where $A_{\mathrm{V}}(2n+1)$ is the number of vertically symmetric $(2n+1) \times (2n+1)$ alternating-sign matrices, and $N_8(2n)$ the number of cyclically symmetric transpose complement plane partitions in a box of size $2n \times 2n \times 2n$. By a well known bijection [16], $A_{\mathrm{VH}}(2L+3)$ is also equal to the total number of vertically and horizontally symmetric fully packed loop (FPL) diagrams on grids of size $2L+3$. An example of a vertically and horizontally symmetric FPL diagram on a grid of size 9 is given in Fig. 2. For later purposes we note that the nonzero entries in the ASM correspond to straight loop segments of length 2.



$$\begin{pmatrix} 0&0&0&0&+&0&0&0&0\\ 0&0&0&+&-&+&0&0&0\\ 0&0&0&0&+&0&0&0&0\\ 0&+&0&0&-&0&0&+&0\\ +&-&+&-&+&-&+&-&+\\ 0&+&0&0&-&0&0&+&0\\ 0&0&0&0&+&0&0&0&0\\ 0&0&0&+&-&+&0&0&0\\ 0&0&0&0&+&0&0&0&0 \end{pmatrix}$$

Figure 2: A vertically and horizontally symmetric FPL diagram on a grid of size 9 and the corresponding alternating-sign matrix.

The vertical and horizontal symmetry constraint enforces certain edges to contain loop segments, and also that the four quadrants of the grid are mirror images of each other. We therefore only have to consider FPL diagrams on an $L \times L$ patch, see Fig. 3. The six FPL patterns that can

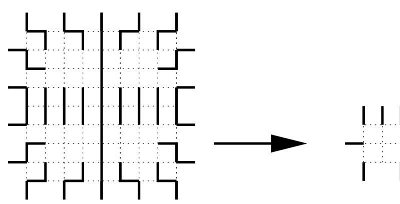


Figure 3: Reduction of a $2L+3$ grid with vertical and horizontal symmetry to an $L \times L$ patch.

be drawn on the 3×3 patch are depicted in Fig. 4. For each FPL diagram one can define a link pattern describing the way the external edges are connected to each other by the loop segments: We number from 1 to L the external edges that contain loop segments on the left and bottom of the $L \times L$ patch, as in the top left diagram of Fig. 4. If edge i is connected to edge j we put an opening parenthesis "(" at i and a closing parenthesis ")" at j. If edge i is connected to one of the external edges on the top we put a closing parenthesis at i. The link patterns in Fig. 4 are thus))), ()) and)() and diagrams with the same link pattern are grouped together.

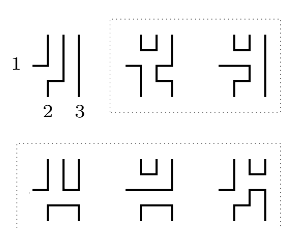


Figure 4: FPL diagrams for $L=3$. Diagrams with the same link pattern are grouped together. The last FPL diagram corresponds to the VHASM given in Fig. 2.

Following Razumov and Stroganov [4] it was further conjectured that not only the sum of

components of $\psi_0(1)$ is related to a counting problem, but also each coefficient of $\psi_0(1)$. Writing

$$\psi_0(1) = \sum_{\pi \in \mathrm{LP}_L^0} c_L(\pi)|\pi\rangle, \tag{10}$$

the conjecture states that $c_L(\pi)$ is equal to the number of FPL diagrams on the $L \times L$ patch with link pattern π. For example for $L = 3$ we obtain

$$\psi_0(1) = (1, 2, 3) \tag{11}$$

on the basis $\{))), ()),)()\}$, indeed corresponding to the enumeration of FPL diagrams with the same link pattern, see Fig. 4.

2.2 A refined conjecture

For general values of the parameter a we find that the conjecture above is refined in the following way. If we write

$$\psi_0(a) = \sum_{\pi \in \mathrm{LP}_L^0} c_L(\pi, a)|\pi\rangle, \tag{12}$$

the coefficient $c_L(\pi, a)$ will be a polynomial in a. We have observed from exact calculations for small values of L that the coefficient of a^j in $c_{2n}(\pi, a)$ (resp. $c_{2n+1}(\pi, a)$) enumerates FPL diagrams on the $L \times L$ patch with connectivity π *and* having $2j$ (resp. $2j + 1$) vertical line segments of length 2 measured from the top. We will illustrate this by two examples.

For $L = 3$ we find

$$\psi_0(a) = (a, 2, 2 + a), \tag{13}$$

on the basis $\{))), ()),)()\}$. Assign to each FPL diagram on the 3×3 patch a weight a^j if it has $2j + 1$ vertical line segments on the first row, as in Fig. 5. The coefficient $c_3(\pi, a)$ of $\psi_0(a)$ is then given by the sum of weighted FPL diagrams having link pattern π.

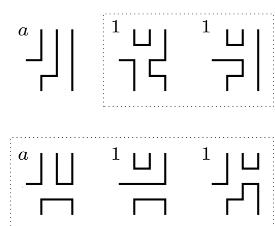


Figure 5: Weighted FPL diagrams for $L = 3$. A diagram with $2j + 1$ vertical line segments in the top row is assigned a weight a^j. Diagrams with the same link pattern are grouped together.

For the second example we look at the even system $L = 4$ and obtain

$$\psi_0(a) = (a^2, 3a(2 + a), 2a(3 + a), 3a, 3(2 + a), 3), \tag{14}$$

on the basis $\{)))),))(),)()), ())), ()(), (())\}$. These components are indeed recovered by the enumeration of weighted FPL diagrams, if we assign to each FPL diagram a weight a^j if it has $2j$ vertical line segments in the top row, as in Fig. 6. The refined conjecture formulated above is similar to but differs from Di Francesco's recent results [9].

2.3 An application

The conjectures above can be used to derive properties of FPL diagrams orASMs. For example, the normalization of the one-boundary ground-state

$$Z^{(1)}(a) = \sum_{\pi \in \mathrm{LP}_L^0} c_L(\pi, a), \tag{15}$$

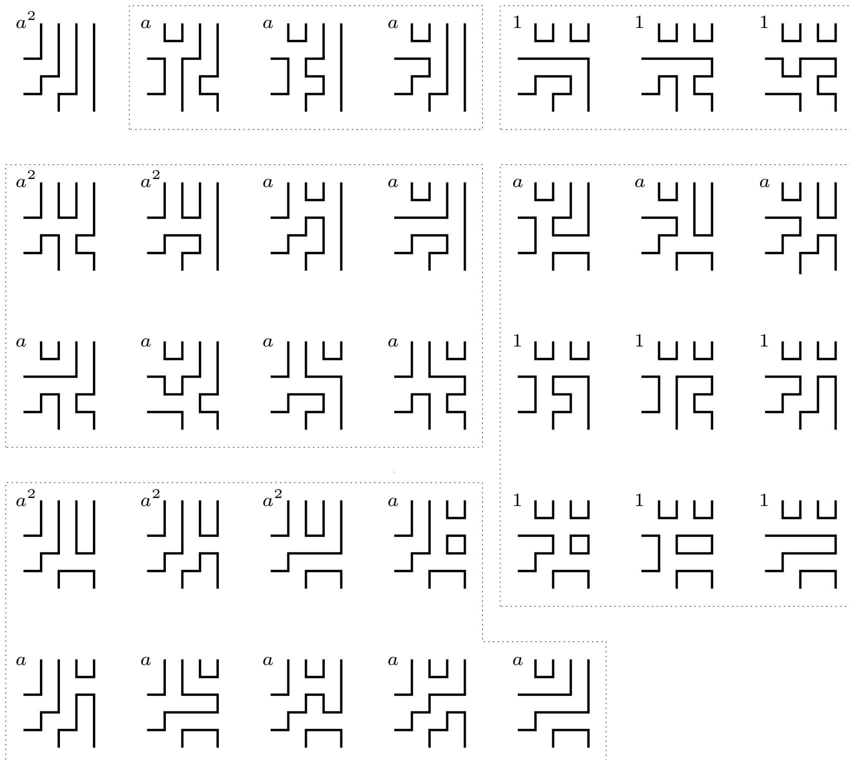


Figure 6: Weighted FPL diagrams for $L = 4$. A diagram with $2j$ vertical line segments in the top row is assigned a weight a^j. Diagrams with the same link pattern are grouped together.

is conjectured to be equal to the partition function of $(2L+3) \times (2L+3)$ VHASMs in which each -1 on the row below the horiontal symmetry axis is given a weight a. The average density $\rho_L(a)$ of -1's on this row is thus given by

$$\rho(a) = \frac{1}{L} \frac{\mathrm{d} \log Z^{(1)}(a)}{\mathrm{d}a}. \tag{16}$$

While we have not found a general expression for $Z^{(1)}(a)$, we have found numerically that

$$\rho_L(1) = \begin{cases} \dfrac{3L+8}{8(2L+3)} & \text{for even } L \\ \dfrac{3(L^2-1)}{8L(2L+3)} & \text{for odd } L \end{cases} \tag{17}$$

In the thermodynamic limit $\rho_\infty(1) = 3/8$ whereas one would expect a value of $1/3$ in a completely random region.

2.4 Relation to the XXZ chain with diagonal boundary conditions

We would like to comment about the spectrum of the Hamiltonian given in (1). It is convenient to parametrize a defined in (1) as follows:

$$a = \frac{3}{1 + 2\cos\delta} > 0. \tag{18}$$

The spectrum of H acting in the link pattern space coincides with that of the Hamiltonian of the XXZ spin 1/2 quantum chain with diagonal boundary conditions and asymmetry parameter $\Delta = -1/2$ [17]:

$$\begin{aligned} H_{\mathrm{D}} &= -\frac{1}{2}\left\{\sum_{j=1}^{L-1}\left(\sigma_j^x\sigma_{j+1}^{\mathrm{x}} + \sigma_j^y\sigma_{j+1}^y - \frac{1}{2}\sigma_j^z\sigma_{j+1}^z\right)\right. \\ &\quad \left. + \frac{\sqrt{3}}{2}\left(\tan\left(\frac{\pi}{6}+\frac{\delta}{2}\right)(\sigma_1^z - 1) + \tan\left(\frac{\pi}{6}-\frac{\delta}{2}\right)(\sigma_L^z - 1)\right)\right\} + \frac{3}{4}(L-1), \end{aligned} \tag{19}$$

where σ^x, σ^y and σ^z are Pauli matrices. For $a \geq 1$ $(0 \leq \delta < 2\pi/3)$ the Hamiltonian H_{D} is Hermitian. Notice that although in (1) the parameter a appears in one boundary generator only, it appears in both boundary terms in the expression for H_{D}. Moreover there exists a similarity transformation which relates H to H_{D}. This last observation is not trivial since Jordan cell structures often appear in stochastic processes and two Hamiltonians might have the same spectrum but might not be related by a similarity transformation [17].

Since H describes a stochastic process it has a positive spectrum with a unique ground-state of zero energy. The same properties are therefore inherited by H_{D}. In the special case $a = 1$ $(\delta = 0)$, using different methods, in [18] it was shown that H_{D} has a positive spectrum and a zero ground-state energy for all system sizes.

We will now shortly comment on the spectrum of H_{D} (or H) in the continuum limit. It is independent of a [19], it is the same for L even and odd and is given by the Gauss model. A compact way to describe the spectrum is to use [20] the sum of the characters of the [1/24] and [3/8] representations of the $c = 1$ Ramond representations of $N = 2$ superconformal field theory [21]. For $a = 0$ the Hamiltonian H_{D} is $U_q(sl(2))$-invariant and the finite-size scaling spectrum is different [22].

3 The two-boundary case

We can extend the Hamiltonian (1) to also include a boundary term at site L,

$$H = a(1 - f_-) + b(1 - f_+) + \sum_{j=1}^{L-1}(1 - e_j). \tag{20}$$

As before, the generators f_- and e_i satisfy the relations of the one-boundary Temperley-Lieb algebra, but with the additional relations

$$\begin{aligned} f_+^2 &= f_+, \\ e_{L-1}f_+e_{L-1} &= e_{L-1}, \\ IJI &= I, \qquad JIJ = J, \end{aligned} \tag{21}$$

where

$$I = \prod_{i=0}^{L/2-1} e_{2i+1}, \qquad J = f_- \prod_{i=1}^{L/2-1} e_{2i}f_+, \qquad \text{for even } L, \tag{22}$$

$$I = f_- \prod_{i=1}^{(L-1)/2} e_{2i}, \qquad J = \prod_{i=1}^{(L-1)/2} e_{2i-1}f_+, \qquad \text{for odd } L. \tag{23}$$

In the link pattern representation, see Fig. 1, loops are now also allowed to be connected to the right boundary of the strip. Connections to the right boundary are facilitated by f_+ in much the same way as they are for the left boundary by f_-.

We wish to study the ground-state of the Hamiltonian (20), which lies in the subspace of link patterns for which all sites are connected. One may furthermore choose not to make a distinction between the left and right boundary, but to identify them, and this is what we will do in the following. As before we denote link patterns by sequences of parentheses and vertical bars. Parentheses are used for sites that are connected to each other while a vertical bar will now stand for sites connected to the boundary. For example, the ground-state sector of the Hamiltonian (20) with identified boundaries for $L = 4$ is given by the linear span of the six link patterns

$$||||, \quad ()||, \quad |()|, \quad ||(), \quad ()(), \quad (()). \tag{24}$$

We will denote the linear span of such link patterns by LP_L^*. Its dimension is given by

$$\dim \mathrm{LP}_L^* = \binom{L}{\lfloor L/2 \rfloor}, \tag{25}$$

which is the same as $\dim \mathrm{LP}_L^0$.

We will now formulate a doubly refined conjecture based on observations for small systems. For our purposes we first need to formulate a new observation: For $a = b = 1$ and *odd* L the ground-state is obtained by counting FPL configurations with appropriate link patterns on an $(L+1) \times L$ patch with boundary conditions as in Fig. 7.[1] We can now formulate a doubly refined

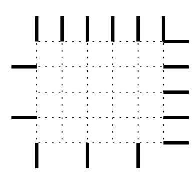


Figure 7: A 6×5 patch corresponding to the ground-state of the Hamiltonian (20) for $L = 5$ and identified boundaries.

conjecture for odd systems. Writing

$$\psi_0(a,b) = \sum_{\pi \in \mathrm{LP}_L^*} c_L(\pi, a, b) |\pi\rangle, \tag{26}$$

the coefficient $c_L(\pi, a, b)$ is a polynomial in a and b. From exact calculations for small values of $L = 2n+1$ we observed that the coefficient of $a^j b^k$ in $c_{2n+1}(\pi, a, b)$ enumerates FPL diagrams on the $(L+1) \times L$ patch with connectivity π *and* having $2j+1$ vertical line segments in the top row and $2k$ horizontal line segments in the last column. For $L = 3$ we find for example

$$\psi_0(a,b) = (a + b + ab, 2 + b, 2 + a), \tag{27}$$

on the basis $\{|||, ()|, |()\}$, indeed corresponding to the weighted enumeration of FPL diagrams, see Fig. 8. In addition we note that the normalization for odd system sizes,

$$Z_{2n+1}^{(2)}(a,b) = \sum_{\pi \in \mathrm{LP}_{2n+1}^*} c_{2n+1}(\pi, a, b), \tag{28}$$

factorises completely,

$$Z_{2n+1}^{(2)}(a,b) = \tilde{Z}_{2n+1}^{(1)}(a) \tilde{Z}_{2n+1}^{(1)}(b). \tag{29}$$

For $a = b = 1$ and *even* L it was conjectured in [11] that the coefficients of the ground-state enumerate FPL configurations with appropriate link patterns on an $(L+1) \times L$ patch with

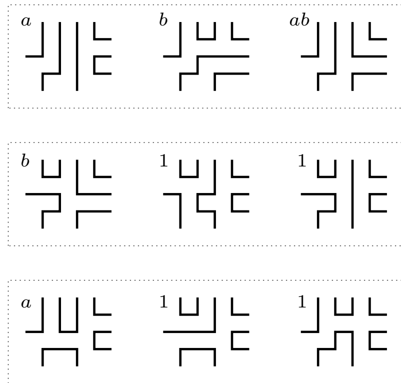


Figure 8: Weighted FPL diagrams for $L = 3$. A diagram with $2j+1$ vertical line segments in the top row and $2k$ horizontal line segments in the rightmost column is assigned a weight $a^j b^k$. Diagrams with the same link pattern are grouped together.

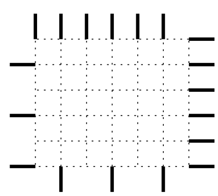


Figure 9: A 7×6 patch corresponding to the ground-state of the Hamiltonian (20) for $L = 6$ and identified boundaries.

boundary conditions as in Fig. 9. The doubly refined conjecture for even systems is formulated as follows. Write the ground-state as

$$\psi_0(a, b) = \sum_{\pi \in \mathrm{LP}^*_L} c_L(\pi, a, b)|\pi\rangle, \tag{30}$$

then the coefficient $c_L(\pi, a, b)$ is a polynomial in a and b. We have observed from exact calculations for small values of $L = 2n$ that the coefficient of $a^j b^k$ in $c_{2n}(\pi, a, b)$ enumerates FPL diagrams on the $(L+1) \times L$ patch with connectivity π *and* having $2j$ or $2j+1$ vertical line segments in the top row and $2k$ horizontal line segment in the last column.

4 Conclusion

This paper contains two conjectures concerning the relation between parameters dependent ground-states of Hamiltonians describing stochastic processes and combinatorial properties of FPLs on grids with special boundary conditions.

We hope that the existence of more conjectures of this kind will help to bring, finally, a proof of the many known parameter-fixed conjectures which are already in the literature (see the references mentioned in the introduction).

In both conjectures the parameters are related to boundary operators and the bookkeeping of the FPLs is also done by looking at how the loops touch the boundaries. This is in contrast to the refined conjecture of Di Francesco [9] (which was a source of inspiration for the present paper) in which the parameter changes the bulk interaction but the bookkeeping of the FPLs still

[1] In [11] it was observed that the ground-state is also obtained by counting FPL configurations with appropriate link patterns on an $L \times (L+1)/2$ patch. Compared to this observation, ours gives rise to an overall factor (equal to $A_{\mathrm{V}}(L+2)$) in each of the components. The relation between both observations is not clear.

counts the way the loops touch the boundary. The second difference between the two approaches can be understood in the following way. If, in analogy with statistical physics, one interprets the parameters dependent ground-state as a partition function depending on fugacities, see e.g. [23] for a justification of this interpretation, our two conjectures give informations on the number of "defects" on the boundary (for example the number of non-zero entries in the VHASM matrices on a row). The refined conjecture of [9] gives the space distribution of one "defect" (the position of 1 in the first row of ASMs).

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